Beam Propagation in Random Media
with Applications to Imaging and Communication

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We are interested in describing (time harmonic) wave beams propagating through a complex medium modeled as a random field. Altough one may be interested in using an incoherent source beam [8] we discuss here the case with a coherent or deterministic source beam. When the beam propagates through the medium it gradually loses is coherence due to scattering. That is, the wave energy is scattered and transferred from the coherent to the incoherent part of the beam. We want to describe this process. We are not interested in describing the wave beam in a particular realization of the random medium, but rather the statistics of the wave field and how it depends on the statistics of the random medium. In fact, we describe the wave statistics via the lower order moments of the field, such moments are typically what is needed to analyze the applications we have in mind which relates to imaging and communication through a complex medium. For instance, in wireless communication when the beam propagates through a complex medium like the turbulent atmosphere, the so called fading and fluctuations of the wave intensity reaching a receiver is fundamental to describe the channel capacity or ability to communicate, see [2].

In order to reach the goal of describing the wave statistics we exploit separations of scales that are present in the problem. The main scales we consider are the central wave-length λ_0, the beam width r_0, the medium coherence length in range ℓ_z and in cross-range ℓ_x and also the total propagation distance L. We then consider the basic beam scaling regime:

$$\lambda_0 \ll \ell_z \sim \ell_x \sim r_0 \ll L.$$

In this scaling regime the back scattering will be very small and we arrive at a description of the forward propagating beam via the ansatz

$$\hat{u}(\omega, z, \mathbf{x}) \sim \frac{ic_0}{2\omega} e^{ikz} \hat{a}(\omega, z, \mathbf{x})$$

where $\hat{u}$ is the solution of the (time harmonic) Helmholtz equation. Note that here we "took out" a rapidly oscillating phase with k the wave number and z the propagation or range direction, so that the wave amplitude $\hat{a}$ oscillates relatively slower in the z direction. In the high frequency scaling limit we then arrive at a description of the wave amplitude in terms of a so called Itô-Schrödinger evolution equation derived in [3]. This is a statistical or "weak description" that can be used to deduce closed equations for all the moments of the harmonic wave field. This equation reads

$$d_z \hat{a} = \frac{1}{2ik} \Delta_\perp \hat{a}\, dz - \frac{k^2 \gamma(\mathbf{0})}{8} \hat{a}\, dz + \frac{ik}{2} \hat{a}\, dB_z$$

with B being a real valued Brownian field with covariance:

$$\mathbb{E}[B_{z_1}(\mathbf{x}_1) B_{z_2}(\mathbf{x}_2)] = \min\{z_1, z_2\} \gamma(\mathbf{x}_1 - \mathbf{x}_2)$$

where

$$\gamma(\mathbf{x}) = \int_{-\infty}^{\infty} \mathbb{E}\left[\mu(0,\mathbf{0})\mu(z,\mathbf{x})\right] dz. \quad \gamma(\mathbf{0}) < \infty.$$

Here μ is the random field giving the random fluctuations in the medium

$$c^{-2}(z,\mathbf{x}) = c_0^{-2} \begin{cases} 1+\mu(z,\mathbf{x}) & \text{if} \quad z \in (0,L), \\ 1 & \text{else} \end{cases}$$

with $\mu(z,\mathbf{x})$ being centered, stationary and with coherence lengths ℓ_x, ℓ_z as mentioned above.

We now ask the question what part of the medium statistics determines the wave field statics. The first moment, or mean field $\mathbb{E}[\hat{a}(\omega,z,\mathbf{x})]$, decays exponentially fast due to scattering of the wave energy on a length scale, the so called scattering mean free path, which is determined by the one medium parameter, the medium range correlation length $\gamma(\mathbf{0})$. We remark that here we take the expectation with respect to the model for the random medium fluctuations. The second moment at range z, $\mathbb{E}[\hat{a}(\omega,z,\mathbf{x})\overline{\hat{a}}(\omega,z,\mathbf{x}+\Delta\mathbf{x})]$, can also be derived explicitly. However, this cross moment depends in general on the full spectrum $\gamma(\mathbf{x})$. In order to describe in particular the fluctuations of the intensity of the transmitted wave field one needs the fourth moment of the wave field. The Itô-Schrödinger equation leads to a transport equation for the fourth moment, but an explicit solution for this is not known. In [5] we show however that an explicit description for the fourth moment can be obtained in a secondary scaling regime, the so called scintillation regime, corresponding the a relatively broad beam or r_0 larger than ℓ_x. In fact, the resulting description corresponds to a quasi Gaussian property in that the fourth moment can be described in terms of the second moment as in the case of a Gaussian random field. The wave description we just outlined is used in particular in [6] to analyze a so called speckle imaging configuration. Here the statistics of the transmitted speckle, a fourth order quantity, is used in a source imaging procedure.

We summarize the complexity of the wave descriptions outlined above. We start out with the Helmholtz equations for the random field in a random medium which is typically prohibitive to solve numerically in the applications we have in mind due to the relatively short wave length and rapid medium fluctuations. Then we identify in the high frequency scaling regime the Itô-Schrödinger equation which may form the basis for feasible numerically simulations via so called phase screen methods. Moreover, this description leads to explicit descriptions for the first two wave moments. The fourth moment derives from a complicated transport equation, a pde with in general eight lateral coordinates in addition to the range coordinate. However, in the so called scintillation regime we arrive at explicit expressions also for the fourth moment, a description that is important in a number of applications.

We next remark on a medium fabric imaging configuration. We assume here an anisotropic medium scaling so that

$$\lambda_0 \sim \ell_z \ll \ell_x \sim r_0 \ll L.$$

In this case the backscattering from the medium will be stronger than in the scale isotropic case described above. In this case the medium parameters that determine the backscattered wave spectrum are in the simplest case

$$\Delta_{\mathbf{x}}\check{\gamma}(0,\mathbf{0}, \quad \check{\gamma}(2k,\mathbf{0}), \quad \Delta_{\mathbf{x}}\check{\gamma}(2k,\mathbf{0}),$$

for

$$\check{\gamma}(k,\kappa) = \int\int \mathbb{E}[\mu(0,\mathbf{0})\mu(z,\mathbf{x})]e^{i(zk+\mathbf{x}\cdot\kappa)}\,dz\,dx.$$

Here the argument $2k$ reflects the coupling between the forward propagating and reflected waves. In general the medium may be only "locally stationary" so that the above parameters vary with respect to range and in [4] we describe how measurement of in particular the (spectral) dynamic width of the backscattered wave energy can be used in an imaging procedure for the changes in the medium statistics with respect to range.

We finally remark that above we discussed relatively long range propagation. It is also of interest to describe the wave corruption over relatively short ranges, for instance for so called "last mile" links in communication applications. In this case different tools need to be used for the wave description and work on such a characterization is in progress [7].

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